

# Graphical representation of generalized quantum measurements

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We present graphical representation for generalized quantum measurements (POVM). We represent POVM elements as Bloch vectors and find the conditions these vectors should satisfy in order to describe realizable physical measurements. We show how to find probability of measurement outcome in a graphical way. The whole formalism is applied to unambiguous discrimination of non-orthogonal quantum states.

## I. INTRODUCTION

Quantum measurement is a link between quantum and classical world. The measurement influence the system that is being measured by changing it to one of the states that measurement apparatus can recognize. Neither Schrödinger equation, nor any relativistic description of quantum mechanics, says anything about the dynamics of measurement. Fortunately, quantum theory can predict the probabilities of measurement outcomes. For our purpose that is enough to say something about the physics behind.

Despite the fact that quantum measurement is probabilistic and "invasive" process it can be found very useful in some interesting and common (i.e. not only laboratory) applications. For example it allows us to gain some knowledge on the system being measured. If someone prepares one of the two known orthogonal quantum states and asks us what the state is. We can answer his question by performing appropriate measurement. On the other hand, destructive property of measurement makes quantum cryptography secure [1]. If somebody wants to get information about the quantum system he has to "touch" it and therefore change it. It means that when someone eavesdrops he leaves "fingerprints" behind him which can be used in the future to detect his presence.

One often assumes that quantum measurement (the so called von Neumann measurement) is usually represented by  $k \leq N$  operators where  $N = \dim(H)$  is the dimension of the Hilbert space. These operators satisfy the following conditions:

$$A_i \geq 0 \quad (1)$$

$$\sum_i A_i = I \quad (2)$$

$$A_i A_j = \delta_{ij} A_i \quad (3)$$

and the probability of obtaining during the measurement the result  $i$  is  $P(i|\rho) = \text{Tr}(A_i \rho)$ , where  $\rho$  is the state of the quantum system on which the measurement is performed. It has been shown however, that there are more possible measurement scenarios than von Neumann orthogonal projective measurements [2, 3]. This type of measurements are called POVM (Positive Operator Valued Measure). The three conditions on POVM elements  $A_i$  appeared to be too much. Only two first are necessary to implement any physical measurement. It may look

strange, because we get use to think of the measurement as of something classical and when we say classical we usually think of some *orthogonal* states. The reason why we can have this counterintuitive operators is that we can add some ancillary system  $B$  then allow it to interact in a prescribed way with our system  $A$  and finally we perform the collective measurement on both systems [4]. Getting rid of the last condition gives us a whole bunch of new possibilities. First of all, we are no longer restricted to only  $N$  projective measurement operators. We can have as many operators as we want as long as they fulfill condition (1) and (2). The first one simply says that the probability of outcome cannot be negative. The second one says that the total probability has to sum up to one. This nonorthogonality of measurement operators allows us to perform probabilistic unambiguous discrimination of quantum states - i.e. we can distinguish, with probability less than one, some quantum states which are in general nonorthogonal [5, 6, 7, 8].

In this paper we give simple graphical description of POVM measurements performed on two level quantum systems (qubits) by representing the POVM elements as vectors inside the Bloch sphere. We show that using this formalism one can easily calculate the best strategy for error-free state discrimination of pure states.

## II. POVM ELEMENTS AS SUB-NORMALIZED QUANTUM STATES

Let us consider two restrictions (1) and (2) on POVM elements. First of all, they have to be positive and therefore hermitian. As a matter of fact, there are some positive and hermitian operators that we know very well - density matrices that describe quantum states. The density matrix possesses one feature that is not necessary for the POVM operators - its diagonal values have to sum up to one. Taking this into account, we may write any POVM operator in the following form

$$A_i = a_i \rho_i, \quad (4)$$

where  $\rho_i$  is some quantum state in the Hilbert space of the system we want to measure and  $a_i$  is real positive constant.

For two level systems one can visualize density matrix as a vector  $\vec{u}$  on or inside the Bloch sphere. We use the

standard formulae [9] to obtain vector coordinates for the Bloch vector

$$u_x = \text{tr}(\sigma_x \rho), \quad (5)$$

$$u_y = \text{tr}(\sigma_y \rho), \quad (6)$$

$$u_z = \text{tr}(\sigma_z \rho), \quad (7)$$

where  $\sigma_k$ ,  $k \in \{x, y, z\}$  is the  $k$ -th Pauli spin matrix. On the other hand we are able to write any density matrix in the form

$$\rho = \frac{I}{2} + \frac{\vec{u} \cdot \vec{\sigma}}{2}. \quad (8)$$

The vector  $\vec{\sigma} = \vec{x}\sigma_x + \vec{y}\sigma_y + \vec{z}\sigma_z$  where  $\vec{x}$ ,  $\vec{y}$  and  $\vec{z}$  are unit vectors.

At this point we should take a closer look at the density matrix  $\rho_i$  that builds operator  $A_i = a_i \rho_i$ . If it is a mixed state we can always decompose it into two (for the two dimensional Hilbert space) orthogonal pure states. The decomposition goes as follows: for the mixed state  $|\vec{v}_i| < 1$  and therefore

$$\rho_i = \frac{I}{2} + b_i \frac{\vec{n}_i \cdot \vec{\sigma}}{2}, \quad (9)$$

where  $\vec{v}_i = b_i \vec{n}_i$  where  $b_i = |\vec{v}_i|$  and  $n_i$  is the unit vector. This can be written as  $\rho_i = c_i \rho_{i1} + d_i \rho_{i2}$ ,  $1 \geq c_i > d_i \geq 0$

$$\rho_{i1} = \frac{I}{2} + \frac{\vec{n}_i \cdot \vec{\sigma}}{2}, \quad (10)$$

$$\rho_{i2} = \frac{I}{2} - \frac{\vec{n}_i \cdot \vec{\sigma}}{2}. \quad (11)$$

It is easy to see that  $c_i + d_i = 1$  and  $c_i - d_i = b_i$ . The graphical representation (see Fig.1) is very simple

$$\vec{v}_i = c_i \vec{n}_i + d_i (-\vec{n}_i). \quad (12)$$

One should notice that  $c_i$  and  $d_i$  are eigenvalues of  $\rho_i$

$$\rho_i = \lambda_{i1} \rho_{i1} + \lambda_{i2} \rho_{i2}. \quad (13)$$

But, using this decomposition we split one POVM *mixed* operator  $A_i$  into two *pure* operators  $A_{i1}$  and  $A_{i2}$

$$\begin{aligned} A_i &= a_i \rho_i = a_i \lambda_{i1} \rho_{i1} + a_i \lambda_{i2} \rho_{i2} = \\ &= a_{i1} \rho_{i1} + a_{i2} \rho_{i2} = A_{i1} + A_{i2}. \end{aligned} \quad (14)$$

The probability that during the measurement of state  $\rho$  the outcome is  $i$  is given by

$$P(i|\rho) = \text{tr}(A_i \rho). \quad (15)$$

For the mixed operator  $A_i$ , we see that

$$\begin{aligned} P(i|\rho) &= \text{tr}(A_i \rho) = \text{tr}((A_{i1} + A_{i2})\rho) = \\ &= \text{tr}(A_{i1} \rho) + \text{tr}(A_{i2} \rho) = P(i1|\rho) + P(i2|\rho). \end{aligned} \quad (16)$$

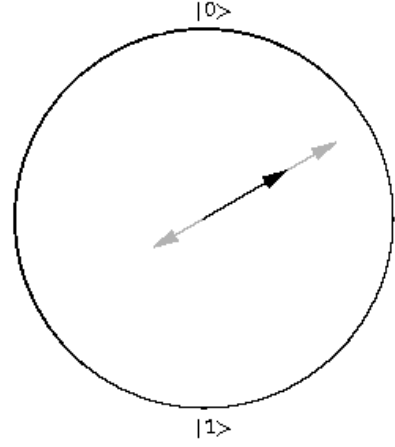


FIG. 1: The decomposition of mixed state (black) into a linear combination of two pure states (gray). It may be written as  $\vec{v}_i = c_i \vec{n}_i + d_i (-\vec{n}_i)$ , where  $c_i$  and  $d_i$  represent the length of gray arrows.

From now we will consider only operators which are of rank one i.e. are proportional to pure states. We do not lose generality because if the operator is of rank two i.e. is proportional to mixed state we can decompose it into two operators of rank one, calculate two probabilities and then add them. We will visualize POVM as

$$\vec{v}_i = \text{tr}(\vec{\sigma} A_i) = a_i \text{tr}(\vec{\sigma} \rho_i) = a_i \vec{n}_i, \quad (17)$$

where  $\vec{n}_i$  is unit vector because  $\rho_i$  is pure state, and quantum states  $\rho$  as

$$\vec{r} = \text{tr}(\vec{\sigma} \rho). \quad (18)$$

Note, that for pure POVM matrix  $\rho_i$  the constant  $a_i$  plays the role of the length of  $\vec{v}_i$ .

Up to now, we have introduced vectors. Right now we will show how to calculate probabilities of measurement outcome using the scalar product  $\vec{v}_i \cdot \vec{r}_j$ . From (8), (17) and (18) we know that

$$A_i = a_i \rho_i = a_i \left( \frac{I}{2} + \frac{\vec{n}_i \cdot \vec{\sigma}}{2} \right), \quad (19)$$

$$\rho = \frac{I}{2} + \frac{\vec{r} \cdot \vec{\sigma}}{2}. \quad (20)$$

Here, it is worth to see the difference between operators  $A_i$  and quantum states. Even if two vectors may look the same in the Bloch sphere picture, they will look different in the notation (19) and (20). Substituting (19) and (20) into (15) one obtains

$$P(i|\rho) = \frac{a_i + \vec{v}_i \cdot \vec{r}}{2}. \quad (21)$$

because only  $I$  and  $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$  have nonzero sum of diagonal elements and contribute to trace.

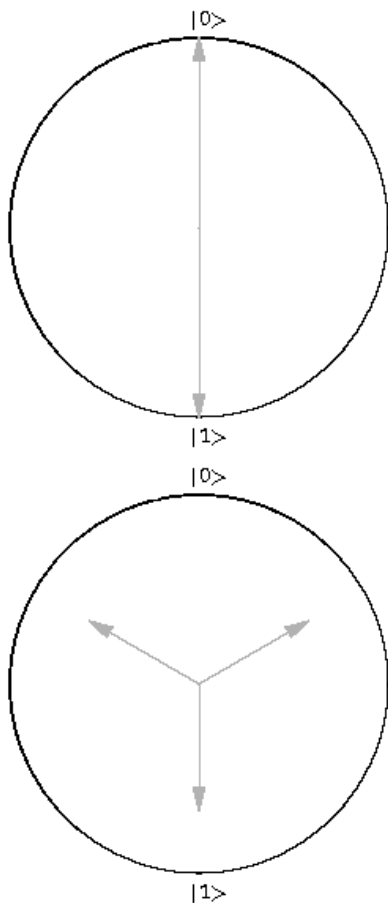


FIG. 2: Without loosing generality, we can consider the Bloch sphere in X-Z plane. Gray arrows represent POVM elements as vectors. Two arrows (top) are just standard von Neumann projectors. Three arrows (bottom) represent more general POVM elements. Note, that the length of the arrows in both cases is different.

Moreover, for pure operators  $A_i$  and pure states  $\rho$  the probability takes the form

$$P(i|\rho) = \frac{a_i(1 + \cos \beta)}{2}, \quad (22)$$

where  $\beta$  is the angle between  $\vec{v}_i$  and  $\vec{r}$ .

Next, we will find the restriction on POVM vectors  $\vec{v}_i$ . If we sum up all of them and take into account (2) we will obtain

$$\sum_i \vec{v}_i = \sum_i \text{tr}(\vec{\sigma} A_i) = \text{tr}(\vec{\sigma} \sum_i A_i) = \text{tr}(\vec{\sigma} I) = 0. \quad (23)$$

The sum of all vectors has to give zero. This is also visible in both cases in the Fig.2.

Now, let us say something about  $a_i$ 's – the lengths of the vectors  $\vec{v}_i$ 's. We can derive one condition that they have to follow by calculating the trace of (2). Trace of the identity operator is equal to the dimension of the Hilbert

space -  $\dim(H)$ . The left hand side gives

$$\text{tr}(\sum_i A_i) = \text{tr}(\sum_i a_i \rho_i) = \sum_i a_i \text{tr}(\rho_i) = \sum_i a_i. \quad (24)$$

so our condition for lengths  $a_i$ 's is

$$\sum_i a_i = \dim(H) = 2, \quad (25)$$

Therefore we get two conditions for the vectors representing POVM

$$\sum_i \vec{v}_i = 0, \quad (26)$$

$$\sum_i |\vec{v}_i| = 2. \quad (27)$$

### III. ERROR-FREE DISCRIMINATION OF QUANTUM STATES

Imagine that somebody sends us one of two pure states  $|\Psi\rangle$  or  $|\Phi\rangle$  with the same probability  $p = \frac{1}{2}$ . We know what these states are, but we do not know which state the sender has chosen. Are we able to tell whether we received  $|\Psi\rangle$  or  $|\Phi\rangle$ ? This case was considered earlier by [5, 6, 7], but we will solve it in a graphical way. It seems that if those states are orthogonal we can do it easily, because this turns out to be purely classical case (even if sometimes the basis differs from  $\{|0\rangle, |1\rangle\}$ ). We simply perform von Neumann projective measurement (see Fig.2 top).

What happens if  $|\Psi\rangle$  and  $|\Phi\rangle$  are not orthogonal? Now, von Neumann measurement is not enough. Although we might set two measurement vectors  $\vec{v}_1$  and  $\vec{v}_2$  of length one ( $a_1 = a_2 = 1$ ) that are parallel (in the Bloch sphere picture) to one of the states, the other state will be lying on the Bloch sphere out of the line given by  $\vec{v}_1$  and  $\vec{v}_2$ . It means, that it may be detected by both  $A_1$  and  $A_2$  and therefore, our answers about identity of the state may be wrong. Let us take  $|\Phi\rangle\langle\Phi| = A_1$ . In order to be more convincing we derive the probability of outcome  $A_i$  while measuring the state  $|\Psi\rangle$ . This probability is given by

$$P(i|\Psi) = \text{tr}(|\Psi\rangle\langle\Psi| A_i) = \frac{1}{2}(1 + \cos \beta_i), \quad (28)$$

where  $\beta_i$  is the angle between the state vector  $\vec{r}_\Psi$  and  $\vec{v}_i$  as before. The vector  $\vec{r}_\Psi$  is lying out of the line given by  $\vec{v}_i$ , so  $\beta_i$  is neither zero nor  $\pi$  and thus we have that both probabilities  $P(1|\Psi)$  and  $P(2|\Psi)$  are nonzero. The consequences are following. If the outcome of the measurement is 2, we know that the only possibility is that the state was  $|\Psi\rangle$ . If the outcome is 1 we know that it probably was  $|\Phi\rangle$ , but there is some chance that it might have been  $|\Psi\rangle$ , so we are not able to distinguish both states without an error.

POVM gives us the possibility of error-free discrimination of quantum states. What we want to obtain is to find two operators  $A_1$  and  $A_2$  that have the following property.  $A_1$  is *sometimes* measured when the first state

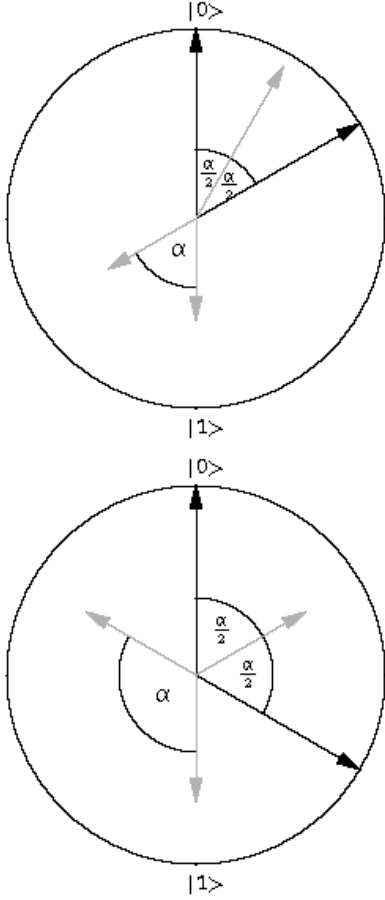


FIG. 3: POVM for two different nonorthogonal states. Black arrows represent two pure states that we want to distinguish  $|0\rangle$  and  $|\Psi\rangle = \cos\frac{\alpha}{2}|0\rangle + \sin\frac{\alpha}{2}|1\rangle$ . Grey arrows represent POVM elements. The arrow lying exactly in between  $|0\rangle$  and  $|\Psi\rangle$  is the error POVM giving no information about which state was sent. Note, that the more orthogonal are the states we try to distinguish the shorter the error POVM arrow becomes.

was sent and is never measured when the second state was sent.  $A_2$  works in the opposite way. It means, we are looking for the vectors  $\vec{v}_1$  and  $\vec{v}_2$  that are antiparallel to the vectors representing the first and the second state

$$\vec{v}_1 \cdot \vec{r}_\Psi = -|\vec{v}_1||\vec{r}_\Psi|, \quad (29)$$

$$\vec{v}_2 \cdot \vec{r}_\Phi = -|\vec{v}_2||\vec{r}_\Phi|. \quad (30)$$

In order to satisfy (26) we need to introduce the third vector. We want "sometimes" to happen quite often, that is why we need  $a_1$  and  $a_2$  to be as large as possible. Moreover, we have no bias in favor of any of the states so we put

$$a_1 = a_2 = a. \quad (31)$$

The probability of successful state discrimination is

$$P_{success} = P(1|\rho_\Phi) = P(2|\rho_\Psi), \quad (32)$$

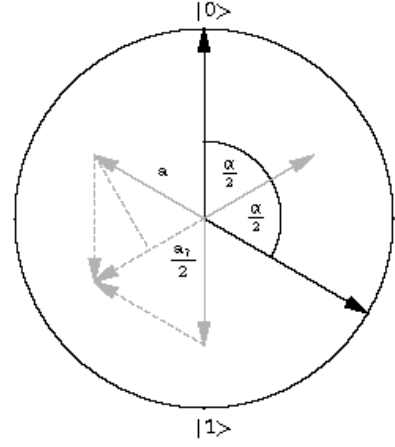


FIG. 4: Graphical way to find the direction and the length  $a_?$  of POVM vector  $\vec{v}_?$ .

where  $\rho_\Phi = |\Phi\rangle\langle\Phi|$  and  $\rho_\Psi = |\Psi\rangle\langle\Psi|$ . The equality sign between two probabilities holds since (29-31) impose symmetry on states and measurement vectors  $\vec{v}_1 \cdot \vec{r}_\Phi = \vec{v}_2 \cdot \vec{r}_\Psi$ . Because of symmetry and condition (26), the third POVM has to point exactly in between  $\vec{r}_\Phi$  and  $\vec{r}_\Psi$  giving

$$\vec{v}_? = -(\vec{v}_1 + \vec{v}_2). \quad (33)$$

Moreover, because of condition (27) we know that

$$a_? = 2(1 - a). \quad (34)$$

The reason why we put quotation sign in the subscript of the third POVM is that it gives completely no information about which state was sent since

$$\vec{v}_? \cdot \vec{r}_\Phi = \vec{v}_? \cdot \vec{r}_\Psi. \quad (35)$$

Let us write explicit formula for (32) keeping in mind that  $A_i$ 's are *pure* operators and the angle between  $\vec{r}_\Phi$  and  $\vec{r}_\Psi$  is  $\alpha$  (see Fig.3)

$$P_{success} = \frac{a}{2}(1 + \cos(\pi - \alpha)) = \frac{a}{2}(1 - \cos \alpha). \quad (36)$$

Then, we go back to (33) and (34) (see also Fig.4) and obtain

$$a_? = 2(1 - a) = 2a \cos \frac{\alpha}{2}. \quad (37)$$

We find that

$$a = \frac{1}{1 + \cos(\frac{\alpha}{2})} \quad (38)$$

and putting this into (36) we get the final formula for the probability of successful state discrimination

$$P_{success} = \frac{1 - \cos(\alpha)}{2(1 + \cos(\frac{\alpha}{2}))}. \quad (39)$$

Moreover, this is the highest possible probability of success because we used as few POVM's as possible maximizing their length  $a$ .

#### IV. CONCLUSIONS

We presented simple and intuitive graphical interpretation of generalized quantum measurements. We have shown that using this method one can easily calculate the best strategy for error-free discrimination of two quantum pure states. Although we have not considered either more than two quantum states or nonuniform probability distribution of states (i.e. some states are more likely to

appear than other states), we believe that this method could be applied as well to such cases.

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